Polygamy of entanglement in multipartite quantum systems

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(Received 20 April 2009; published 4 August 2009)

We show that bipartite entanglement distribution (or entanglement of assistance) in multipartite quantum systems is by nature polygamous. We first provide an analytical upper bound for the concurrence of assistance in bipartite quantum systems and derive a polygamy inequality of multipartite entanglement in arbitrary-dimensional quantum systems.

DOI: 10.1103/PhysRevA.80.022302 PACS number(s): 03.67.Mn, 03.65.Ud

Whereas quantum entanglement in bipartite quantum system has been intensively studied with various applications, entanglement in multipartite quantum systems still seems far from our understanding. One of the most distinct phenomena of quantum entanglement in multiparty systems is that it cannot be freely shared among parties. For example, if two parties share a maximally entangled state, they cannot have entanglement or even classical correlations with any other parties. This is known as the monogamy of entanglement (MoE): the entanglement between one party and all the others in multipartite quantum systems bounds the sum of entanglement between one and each of the others. MoE was shown to have a mathematical characterization in forms of inequalities in multiqubit systems [1,2] using concurrence [3] to quantify the shared entanglement among subsystems.

However, monogamy inequality using concurrence is known to fail in its generalization for higher-dimensional quantum systems. In other words, the existence of quantum states violating concurrence-based monogamy inequality was shown in higher-dimensional systems [4,5]. Later, it was shown that those counterexamples of concurrence-based monogamy inequality still show monogamous property of entanglement by using a different entanglement measure [5], and this exposes the importance of having a proper way of quantifying entanglement.

Whereas, monogamy inequality is about the restricted sharability of multipartite entanglement, entanglement distribution, which can be considered as a dual concept to the sharable entanglement, is known to have a polygamous property (sometimes referred as dual monogamy) in multipartite quantum systems. A mathematical characterization for the polygamy of entanglement (PoE) was first provided for multiqubit systems [6,7] using concurrence of assistance (CoA) [8]. Recently, polygamy inequality was also shown in tripartite quantum system of arbitrary dimension using entanglement of assistance for the quantification of entanglement distribution [9]. However, a general polygamy inequality of entanglement in multipartite higher-dimensional quantum system is still an open question.

Here, we provide a strong clue for this question. We provide an analytical upper bound of CoA for arbitrary bipartite mixed states and derive a polygamy inequality of multipartite entanglement in arbitrary-dimensional quantum systems.

The upper bound is saturated for any two-qubit states, and thus the derived polygamy inequality coincides with the one proposed in [7] for multiqubit systems.

For any bipartite pure state $|\phi\rangle_{AB}$, its concurrence $C(\langle \phi \rangle_{AB})$ is defined as [3]

$$C(\langle \phi \rangle_{AB}) = \sqrt{2(1 - \text{tr} \rho_A^2)},$$

where $\rho_A = \text{tr}_B(\langle \phi \rangle_{AB} \langle \phi \rangle)$. For any mixed state $\rho_{AB}$, its concurrence is defined as

$$C(\rho_{AB}) = \min_k \sum_{l} p_k C(\langle \phi_l \rangle_{AB}),$$

and its CoA is

$$C^\text{CoA}(\rho_{AB}) = \max_k \sum_{l} p_k C^\text{CoA}(\langle \phi_l \rangle_{AB}),$$

where the minimum and the maximum are taken over all possible pure state decompositions, $\rho_{AB} = \sum_k p_k |\phi_k\rangle_{AB} \langle \phi_k|$. For a three-qubit state $|\phi\rangle_{ABC}$, a polygamy inequality of entanglement was first introduced as [6]

$$C^\text{poly}(\langle \phi \rangle_{ABC}) \leq (C^\text{CoA}_{AB})^2 + (C^\text{CoA}_{AC})^2,$$

where $C^\text{CoA}_{AB} = C^\text{CoA}(\langle \phi \rangle_{AB})$ is the concurrence of a three-qubit state $\langle \phi \rangle_{AB}$ for a bipartite cut of subsystems between $A$ and $B$, and $C^\text{CoA}_{AC} = C^\text{CoA}(\langle \phi \rangle_{AC})$ with $\rho_{AC} = \text{tr}_B(\langle \phi \rangle_{ABC} \langle \phi \rangle)$. Later, a generalization of Eq. (3) into $n$-qubit systems [7]

$$C_{A_1A_2\cdots A_k}^\text{CoA} \leq (C_{A_1A_2})^2 + \cdots + (C_{A_{k-1}A_k})^2$$

was also introduced for an arbitrary $n$-qubit pure state $|\phi\rangle_{A_1\cdots A_n} \in (C^2)^{n^n}$.

Now, let us consider a bipartite pure state of arbitrary dimension $|\phi\rangle_{AB} = \sum_{i,j} \sum_{a_{ij}a_{i\ell}a_{jkl}} |i\ell\rangle_{AB} \in \mathcal{H}^A \otimes \mathcal{H}^B = \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$. In terms of the coefficients of $|\phi\rangle_{AB}$, its concurrence can also be expressed as [10]

$$C(\langle \phi \rangle_{AB}) = 2(1 - \text{tr} \rho_A^2) = 4 \sum_{i,j} \sum_{k<l} |a_{ij} a_{jkl} - a_{i\ell} a_{i\ell}|^2,$$

where $\rho_A = \text{tr}_B(\langle \phi \rangle_{AB} \langle \phi \rangle)$.

Let $m$ and $n$ be ordered pairs such that

$$m = (i,j), \quad n = (k,l), \quad i < j, \quad k < l,$$

with $i,j = 1, \ldots, d_1$, $k,l = 1, \ldots, d_2$. As $m$ has $D_1 = d_1(d_1 - 1)/2$ choices of taking $i$ and $j$ from $d_1$ elements, and similarly $D_2 = d_2(d_2 - 1)/2$ choices for $n$, with some appropriate

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orderings of \((i,j)\) and \((k,l)\), we can label \(m\) and \(n\) as
\begin{equation}
 m = 1, \ldots, D_1, \quad n = 1, \ldots, D_2.
\end{equation}
We also let
\begin{equation}
 L^n_A = P^n_A(\ketbra{i}{j}_A) P^n_A, \\
 L^n_B = P^n_B(\ketbra{k}{l}_B) P^n_B,
\end{equation}
where \(P^n_A = \ketbra{i}{j}_A \) and \(P^n_B = \ketbra{k}{l}_B \) are the projections onto the subspaces spanned by \(\{\ket{i}_A, \ket{j}_A\} \) and \(\{\ket{k}_B, \ket{l}_B\} \), respectively. By straightforward calculation, we have
\begin{equation}
\braket{\psi(L^n_A \otimes L^n_B) \psi^*}^2 = \left| \sum_{m,n} |\langle \psi | L^n_A \otimes L^n_B | \psi \rangle |^2 \right|.
\end{equation}
and together with Eq. (6), we have
\begin{equation}
 C^2(\rho_{AB}) = \sum_{m=1}^{D_1} \sum_{n=1}^{D_2} \braket{\psi(L^n_A \otimes L^n_B) \psi^*}^2.
\end{equation}

Equation (10) can be considered as the squared concurrence of the pure state (possibly un-normalized)
\begin{equation}
(P^n_A \otimes P^n_B) |\psi_{AB}\rangle = a_{ik} |ik\rangle_A + a_{il} |il\rangle_A + a_{jk} |jk\rangle_A + a_{jl} |jl\rangle_A
\end{equation}
in two-dimensional subspaces of \( \mathcal{H}_A \) and \( \mathcal{H}_B \) spanned by \(\{\ket{i}_A, \ket{j}_A\} \) and \(\{\ket{k}_B, \ket{l}_B\} \), respectively. Furthermore, Eq. (11) implies that the concurrence of a bipartite pure state \(|\psi_{AB}\rangle\) can be decomposed into the concurrences of two-qubit subspaces in Eq. (10).

For a mixed state \( \rho_{AB} = \sum_{\xi} \rho_{\xi} |\xi\rangle \langle \xi| \) with \( |\xi\rangle \langle \xi|_{AB} = \rho_{\xi} |\xi\rangle \langle \xi| \) and \( |\xi\rangle |\xi\rangle_A \) to \( \rho_{\xi} \), its average concurrence is
\begin{equation}
\sum_i p_i C(\psi_i) = \left( \sum_{m,n} \braket{\psi | L^n_A \otimes L^n_B | \psi}^2 \right)^{1/2} = \left( \sum_{m,n} \braket{\xi | L^n_A \otimes L^n_B | \xi}^2 \right)^{1/2}
\end{equation}
and thus its CoA is
\begin{equation}
 C^2(\rho_{AB}) = \max_i p_i C(\psi_i) \leq \max_i \sum_{m,n} \braket{\xi | L^n_A \otimes L^n_B | \xi}^2
\end{equation}
where the maxima are taken over all possible pure state decompositions of \( \rho_{AB} \). Here, we note that the term after the maximum in the last line of Eq. (14) is the average concurrence of the state (possibly un-normalized)
\begin{equation}
 (\rho_{AB})_{mn} = (P^n_A \otimes P^n_B) \rho_{AB} (P^n_A \otimes P^n_B)
\end{equation}
in the two-qubit subspace spanned by \(\{\ket{i}_A, \ket{j}_A\} \) and \(\{\ket{k}_B, \ket{l}_B\} \). Thus, the maximum value can be considered as the CoA of the two-qubit state \( (\rho_{AB})_{mn} \); therefore, by the optimization methods for CoA in two-qubit systems [8], we have
\begin{equation}
 C^2((\rho_{AB})_{mn}) = \max_i \sum_{m,n} \braket{\xi | (L^n_A \otimes L^n_B) | \xi}^2 = \mathcal{F}(\rho_{AB}, (\rho_{AB})_{mn}).
\end{equation}
where \( (\rho_{AB})_{mn} = (L^n_A \otimes L^n_B) \rho_{AB} (L^n_A \otimes L^n_B) \) and \( \mathcal{F}(\rho_{AB}, (\rho_{AB})_{mn}) \) is the fidelity of \( \rho_{AB} \) and \( (\rho_{AB})_{mn} \) defined as
\begin{equation}
 \mathcal{F}(\rho_{AB}, (\tilde{\rho}_{AB})_{mn}) = \text{tr} \sqrt{\sqrt{\rho_{AB}} (\rho_{AB})_{mn} \sqrt{\rho_{AB}}},
\end{equation}
Now, we are ready to have the following theorem.

Theorem 1. For any bipartite state \( \rho_{AB} \in B(\mathcal{C}^2 \otimes \mathcal{C}^2) \),
\begin{equation}
 C^2(\rho_{AB}) \leq \sum_{m=1}^{D_1} \sum_{n=1}^{D_2} \mathcal{F}(\rho_{AB}, (\rho_{AB})_{mn}) = \tau^2(\rho_{AB}),
\end{equation}
where \( D_1 = d_A d_i^{1-1/2}, \) \( D_2 = d_B d_2^{1-1/2}, \) \( (\rho_{AB})_{mn} = (L^n_A \otimes L^n_B) \rho_{AB} (L^n_A \otimes L^n_B) \), \( \mathcal{C}^2((\rho_{AB})_{mn}) \) is the CoA of \( (\rho_{AB})_{mn} \) to \( \rho_{AB} \).

For any bipartite mixed state \( \rho_{AB} \) of arbitrary dimension, the sum of concurrences of all possible two-qubit subspaces is known to provide a lower bound of the concurrence \( C(\rho_{AB}) \) [11]. By using the lower bound of the concurrence, it was also shown that there is a proper monogamy inequality of entanglement in multipartite quantum systems [11]. Theorem 1 implies that the sum of CoA of \( (\rho_{AB})_{mn} \) from all possible two-qubit subspaces of \( \rho_{AB} \) forms an upper bound of \( C(\rho_{AB}) \). It can be directly checked that this bound is saturated for any pure and two-qubit mixed states.

Now, we provide a polygamy inequality of multipartite entanglement of arbitrary dimension in terms of the upper bound proposed in Theorem 1.

Theorem 2. For any multipartite pure state \( |\psi\rangle_{A_1 \cdots A_n} \) in \( \mathcal{C}^{d_1} \otimes \cdots \otimes \mathcal{C}^{d_n} \),
\begin{equation}
 (\tau^2_{A_1 A_2 \cdots A_k})^2 \leq \tau^2_{A_1 A_2} + \cdots + \tau^2_{A_1 A_k}.
\end{equation}
where \( \tau^2_{A_1 A_2 \cdots A_k} = \tau^2(|\psi\rangle_{A_1 A_2 \cdots A_k}) \) with respect to the bipartite cut \( A_1 - A_2 \cdots A_k \), \( \tau^2_{A_1 A_k} = \tau^2(|\psi\rangle_{A_1 A_k}) \), and \( \rho_{A_1 A_k} \) is the reduced density matrix of \( |\psi\rangle_{A_1 \cdots A_k} \) onto subsystem \( A_1 A_k \) for \( k = 2, \ldots, n \).

Proof. For \( k = 1, \ldots, n \), let \( m_k = (i_k, j_k) \) be an ordered pair of \( i_k, j_k \in \{1, \ldots, d_k\} \) such that \( i_k < j_k \), and let \( M = (m_2, \ldots, m_n) \) be an \( (n-1) \)-tuple of the ordered pairs. By letting \( D_k = d_k(d_k - 1)/2 \) for \( k = 1, \ldots, n \), we have
\begin{equation}
 (\tau^2_{A_1 A_2 \cdots A_k})^2 \leq \mathcal{C}^2_{A_1 A_2 \cdots A_k} = \sum_{m_1} \sum_{M=1}^{D_k} \left( \sum_{M=1}^{D_k} (\mathcal{C}^2_{A_1 A_2 \cdots A_k})_{m,M} \right)^2,
\end{equation}
where \( \mathcal{C}^2_{A_1 A_2 \cdots A_k} \) is the concurrence of the (un-normalized) state \( \langle \psi\rangle_{A_1 A_2 \cdots A_k} \) in the subspace spanned by \( \{\ket{i_1}_A, \ket{j_1}_A\}, \ldots, \{\ket{i_n}_A, \ket{j_n}_A\} \).
As \((|\psi\rangle_{A_1 A_2 \cdots A_n})_{m,M}\) is an \(n\)-qubit (un-normalized) state for each \(m_1\) and \(M\), it satisfies the multiqubit polygamy inequality in Eq. (5), that is,

\[
[(C_{A_1 A_2})_{m_1 m_2}]^2 + \cdots + [(C_{A_1 A_k})_{m_1 m_k}]^2
\]

where \((C_{A_1 A_k})_{m_1 m_k}\) is CoA of \((\rho_{A_1 A_k})_{m_1 m_k}\), the reduced operator of \((|\psi\rangle_{A_1 A_k \cdots A_k})_{m,M}\) onto subsystems \(A_1 A_k\) for \(k=2, \ldots, n\).

From Eqs. (21) and (22), we have

\[
\left(\frac{\partial}{\partial A_1 A_2 \cdots A_n}\right)^2 \leq \sum_{m_1=1}^{D_1} \sum_{m_2=1}^{D_2} \left[\frac{(C_{A_1 A_2})_{m_1 m_2}}{2}\right]^2 + \cdots + \sum_{m_k=1}^{D_k} \sum_{m_k=1}^{D_k} \left[\frac{(C_{A_1 A_k})_{m_1 m_k}}{2}\right]^2
\]

where the last equation is due to Eq. (17) and the definition of \(\frac{\partial}{\partial A_1 A_2 \cdots A_n}\).

As the upper bound of CoA in Theorem 1 is saturated for any two-qubit mixed state, Eq. (20) in Theorem 2 is reduced to Eq. (5) for the case of multiqubit systems. Moreover, it can be easily seen that Eq. (20) is saturated by \(n\)-qubit generalized W-class states [12], that is,

\[
|W\rangle_{A_1 \cdots A_n} = a_1|1 \cdots 0\rangle_{A_1 \cdots A_n} + \cdots + a_n|0 \cdots 1\rangle_{A_1 \cdots A_n},
\]

with \(\sum_{i=1}^{n} |a_i|^2 = 1\).

To summarize, we have shown the polygamous nature of distributed entanglement in multipartite quantum systems of arbitrary dimension. By providing an analytical upper bound of CoA for arbitrary bipartite quantum states, we have derived a polygamy inequality of entanglement in terms of the upper bound. This upper bound is saturated for any two-qubit state, and thus the polygamy inequality proposed here can be considered as a generalization of the result in [7] into higher-dimensional quantum systems.

One of the main difficulties in the study of multipartite entanglement is that there can be several inequivalent classes that are not convertible to each other under stochastic local operations and classical communications (SLOCCs) [13]. These inequivalent classes make us hardly have a universal way of quantifying multipartite entanglement, even in an abstract sense.

However, the existence of inequivalent classes of multipartite entanglement also reveals the different characters among different classes. For example, three-qubit systems are known to have two inequivalent classes of genuine tripartite entanglement: one is the Greenberger-Horne-Zeilinger (GHZ) class [14] and the other one is the W class [13]. Although the inequivalence of the classes is due to SLOCC convertibility [13], these two classes also show extreme differences in terms of monogamy or polygamy inequality of entanglement. In other words, monogamy and polygamy inequalities are saturated by W-class states, whereas the differences between terms in the inequalities can assume their largest values for the GHZ-class state. Thus, MoE and PoE are not only just distinct phenomena in multipartite quantum systems, but they also provide us an efficient way of qualifying multipartite entanglement.

Our result is a case where the polygamy nature of multipartite entanglement in arbitrary-dimensional quantum systems is discussed with mathematical characterizations. Noting the importance of the study on high-dimensional multipartite entanglement, our result can provide a reference for future work on the study of multipartite entanglement.

This work was supported by iCORE, MITACS, and USARO.