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REFERENCES


On the Capacity Region of a Discrete Two-User Channel for Strong Interference

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Abstract—The capacity region of a simple discrete two-user channel for the transmission of two separate messages is studied in the strong interference case. It is shown that in this case the inner bound to the capacity region practically coincides with the outer bound.

I. INTRODUCTION

A two-user channel, also referred to as an interference channel, is one with two input terminals and two output terminals. In this paper the problem of determining the capacity region of such a channel is considered when two source-user pairs utilize the channel to transmit two separate messages simultaneously. This problem was first studied by Ahlswede [1] and was called the \((P, T,2_2, I)\) problem. Since then it has been studied by several researchers [2]-[5]. It is also called the single interference channel problem in the recent survey article by van der Meulen [6]. The reader should consult [2] or the article by van der Meulen for an exact statement of the problem and for definitions of several terms used in this paper, e.g., "achievable rate pair," "achievable region," "capacity region," etc. The capacity region of the problem is denoted by \(G\) in this paper.

Recently Carleial [3] has shown that, for the Gaussian channel with strong interference, each message can be recovered reliably at the same rates that are achievable in the absence of interference. The purpose of this paper is to show the corresponding phenomenon for the discrete channel. By examining a simple but hopefully typical example of a discrete two-user channel, we find an inner bound and an outer bound to the capacity region that are nearly coincident when the interference is very strong.

Let us briefly use the concept of a jointly achievable rate pair in order to explain the inner bound to the capacity region. For this purpose, we also consider the two-user channel with two inputs and two outputs where two independent messages are transmitted from the inputs. In this case, however, it is required that each user must receive both messages. Let us call a rate pair jointly achievable if the pair is achievable in the above situation. The corresponding capacity region is called the jointly achievable capacity region and is denoted by \(G\). This problem was called \((P, T,2, II)\) by Ahlswede [7] who obtained \(G\). This case is also called the compound interference channel by van der Meulen [6]. The region \(G\) is obviously an inner bound to the capacity region \(G\) of our original problem.

The two-user channel considered throughout this paper is a discrete memoryless symmetric channel with two binary inputs and two binary outputs. Its marginal conditional probabilities \(p_i(y_1|x_1,x_2)\) and \(p_i(y_2|x_1,x_2)\) take the following forms:

\[
p_i(0|0,0) = p_i(1|1,1) = 1,
\]

\[
p_i(1|0,0) = p_i(0|1,1) = 0, \quad \text{for } i = 1,2,
\]

\[
p_i(0,0,1) - p_i(1|1,0) = p_i(1|0,1) = p_i(0|0,0) = \bar{s}
\]

\[
p_i(1|0,1) = p_i(0|1,1) = p_i(0|0,0) = s (1)
\]

where \(s = 1 - \bar{s}\) and \(s\) is a parameter representing the degree of interference, with \(0 < s < 1\). We sometimes call this channel a coincidence channel since errorless transmission is possible when the two inputs \(x_1\) and \(x_2\) are coincident.

II. AN OUTER BOUND AND A JOINTLY ACHIEVABLE CAPACITY REGION

In this section, we first introduce expressions for an outer bound \(G\) to the capacity region \(G\). Then we compare \(G\) with the inner bound \(\hat{G}\), and obtain a condition for these two regions to coincide.

The following region \(G\) in the \(R_1, R_2\) plane is known to be an outer bound to the capacity region \(G\) [2, 4]:

\[
G_0 = \co \left\{ \bigcup_{\eta_0, \xi_0} g_0 \right\},
\]

where

\[
g_0 = \{ R_1, R_2 \mid 0 < R_1 < \xi_2, 0 < R_2 < \eta_2 \},
\]

\[
\eta_2 = \I(X_2; Y_2 | X_1),
\]

\[
\xi_2 = \I(X_1; Y_2 | X_2),
\]

and \(\co (A) = \text{convex hull of } A\). The union in (2) is over probability distributions \(Q_1\) and \(Q_2\) on the inputs \(x_1\) and \(x_2\) which are mutually independent. A tighter outer bound \(G^*\) was found by the author [2], but the bound \(G_0\) is sufficient for our problem.

The jointly achievable capacity region \(G\) was obtained by van der Meulen [7]:

\[
G = \co \left\{ \bigcup_{\eta_0, \xi_0} g_0 \right\},
\]

where

\[
g_0 = \{ R_1, R_2 \mid 0 < R_1 < \min (\xi_2, \xi_1), 0 < R_2 < \min (\eta_2, \eta_3) \},
\]

\[
\eta_3 = \I(X_2; Y_2 | X_1),
\]

\[
\xi_3 = \I(X_1; Y_2 | X_2),
\]

\[
\xi_1 = \I(X_1; Y_1 | X_2),
\]

\[
\eta_1 = \I(X_2; Y_1 | X_1).
\]

Let us further introduce the mutual information:

\[
\xi_0 = \I(X_1; Y_1),
\]

\[
\eta_0 = \I(X_2; Y_2),
\]

\[
\xi_1 = \I(X_1; Y_2),
\]

\[
\eta_1 = \I(X_2; Y_1).
\]

The inequalities \(\xi_0 < \xi_1\), \(\eta_0 < \eta_1\), \(\xi_3 < \xi_2\), and \(\eta_3 < \eta_2\) hold for independent inputs, and the region \(G\) is an intersection of two pentagons, as depicted in Fig. 1. Each pentagon is simply the capacity region of a multiple access channel [1], [8] with two inputs \(x_1\) and \(x_2\) but with a single output \(y_1\) or \(y_2\).

Now it is easily seen from Fig. 1 that \(g_0\) and \(g_1\) coincide if and only if

\[
\xi_2 < \xi_1 \text{ and } \eta_2 < \eta_1
\]
hold simultaneously. Therefore, if the above inequalities hold for all input distributions $Q_1, Q_2$, then $G_0$ and $G_1$ coincide with the capacity region $G$. This is a sufficient condition for the coincidence of the bounds, but may not be necessary.

In the case of a Gaussian two-user channel, the condition obtained by Carleial [3] can be obtained from (9) using Gaussian inputs.

III. RESULTS AND DISCUSSIONS

We shall consider the coincidence channel with conditional probabilities (1). Let the input probabilities $Q_1(x_1)$ and $Q_2(x_2)$ be $q_1, \bar{q}_1 = 1 - q_1,$ and $q_2, \bar{q}_2 = 1 - q_2$ for $x_1$ and $x_2 = 0, 1,$ respectively. Then the necessary mutual information is calculated as follows:

$$
\begin{align*}
\xi_1 &= I(X_1; Y_2) = h(sq_1 + \bar{q}_1) - q_1 h(\bar{q}_2) - \bar{q}_1 h(q_2) \\
\eta_1 &= I(X_2; Y_1) = h(sq_2 + \bar{q}_2) - q_2 h(\bar{q}_1) - \bar{q}_2 h(q_1) \\
\xi_2 &= I(X_1; Y_1|X_2) = q_2 h(\bar{q}_1) + \bar{q}_2 h(q_1) - (q_1 + q_2) h(s) \\
\eta_2 &= I(X_2; Y_1|X_1) = q_1 h(\bar{q}_2) + \bar{q}_1 h(q_2) - (q_1 + q_2) h(s) \\
\xi_3 &= I(X_2; Y_1|X_1) = q_1 h(\bar{q}_2) + \bar{q}_1 h(q_2) - (q_1 + q_2) h(s) \\
\eta_3 &= I(X_1; Y_2|X_2) = q_2 h(\bar{q}_1) + \bar{q}_2 h(q_1) - (q_1 + q_2) h(s)
\end{align*}
$$

where $q_1 \cdot q_2 = q_1 q_2 + \bar{q}_1 \bar{q}_2$ and $h(\cdot)$ is binary entropy.

We shall consider only the case $s > 0.5$. In this case it can be shown that $G_f$ includes other known achievable regions introduced in [2], and $G_f$ is a good approximation to $G$. We can first show that the maximum values $r_0$ and $r_f$ on $G_0$ and $G_f$ occur on the $R_1$ axis and that they are equal. The conditional mutual information $\xi_2 = I(X_1; Y_1|X_2)$ takes its maximum value when a single letter $x_2$ of the alphabet $X_2$ is used, for example when we set $q_2 = 0$. Thus,

$$
r_0 = \max_{q_1, q_2} \xi_2(q_1, q_2) = \max_{q_1} \eta_2(q_1, 0)
$$

Let the $q_1$ that maximizes $\xi_2(q_1, 0)$ be denoted by $q_{m}$. We see that $\eta_2 = 0$ for $q_2 = 0$ from (10), and that this maximum value $r_0$ occurs on the $R_1$ axis. Similarly, if we let the maximum value of $\xi_3$ be $r_3$, we find

$$
r_3 = \max_{q_1, q_2} \xi_3(q_1, q_2) = \max_{q_1} \eta_3(q_1, 0)
$$

We have $\xi_2(q_1, 0) > \xi_2(q_1, 0)$ for any $q_1$, since $h(sq_1) > h(\bar{q}_1)$ for $s > 0.5$. Therefore

$$
r_f = \max_{q_1, q_2} (\xi_2, \eta_2)
$$

is reached for $(q_m, 0)$, and we can conclude that $r_f = r_0$ for $s > 0.5$. For $q_2 = 0, \eta_2 = 0$, and again this largest value occurs on the $R_1$ axis.

A sketch of $G_0$ is shown in Fig. 2. We have just shown that the boundary of $G_f$ also meets the $R_1$ axis at $B_1$. It is easily seen that $G_0$ and $G_f$, as well as $G$, are symmetric about the line $R_1 = R_2$ so that in general it is sufficient to describe the shape of the regions for $R_1 > R_2$.

Let us now explain the behavior of the boundary curve of $G_0$. We can show that $\xi_2 + \eta_2$ reaches a maximum for $q_1 = q_2 = 1/2$, with $\xi_2 = \eta_2$ (see point $A$ in Fig. 2). In Fig. 3 the probabilities $q_1$ and $q_2$ are plotted along the axes. It is sufficient to consider the region of the probability plane for which $q_1 + q_2 < 1$ by the symmetry of the problem. $A'$ in Fig. 3 is the point $q_1 = q_m$ and $q_2 = 0$, for which $B_1$ in Fig. 2 is calculated. The boundary curve $AB_1$ of $G_0$ is realized when the point $(q_1, q_2)$ in Fig. 3 moves from $A'$ to $B_1$ along some curve. Similarly the boundary $AB_2$ of $G_0$ is generated by $(q_1, q_2)$ on the
curve \(A'B;\) in the probability plane. For \(s > 0.5\), we can see by numerical calculation that the curve \(A'B;\) is very close to the straight line \(A'B;\). Also, the boundary curve \(AB;\) of \(G_J\), although convex outward, is well approximated by the straight line \(A'B;\).

Now let us examine condition (9) in order to compare \(G_J\) with \(G_O\). Let us define

\[
f(q_1, q_2) = \xi_1 - \xi_2 = h(\tilde{q}_1 + \tilde{s}_q) - q_1 h(\tilde{s}_q) - q_2 h(\tilde{q}_2) - q_1 q_2 h(s).
\]

For \(s = 0.5\), \(f(q_1, q_2)\) is nonpositive for all \(q_1, q_2\). For \(q_1 = q_2 = 1/2\), \(f(1/2, 1/2) = -2 h(s/2) + h(s)/2\) is an increasing function of \(s\) and becomes 0 for \(s = s_0 = 0.5754\). For this value of \(s\), condition (9) is satisfied only at \(A', B_1\), and \(B_2\); \(\xi_1 - \xi_2 = f(q_1, q_2)\) is negative on the curve \(A'B;\) (except at the end points), and \(\eta_1 - \eta_2 = f(q_1, q_2)\) is negative on the curve \(A'B;\). Thus the boundaries of \(G_J\) and \(G_O\) coincide only at \(A, B_1, B_2\) and \(s = s_0\). Since the boundary \(AB;\) of \(G_J\) is well approximated by the straight line \(AB;\), \(G_J\) will be a fairly good approximation to \(G\) even in this case.

When \(s\) exceeds \(s_0\), the region in the probability plane for which \(\xi_1 < \xi_2\) moves towards the \(q_1\) axis. This region and the region for which \(\eta_1 < \eta_2\) are shaded in Fig. 3. The points \(C_1\) and \(C_2\) in Fig. 3 are on the boundaries of the above regions, and the points \(C_1\) and \(C_2\) in Fig. 2 are generated by these probabilities, respectively. Thus the boundaries of \(G_J\) and \(G_O\) coincide only at \(A, B_1, B_2\) for \(s = s_0\). Since the boundary \(AB;\) of \(G_O\) is well approximated by the straight line \(AB;\), \(G_O\) will be a fairly good approximation to \(G\) even in this case.

We can further show that the segments \(B_1C_1\) and \(B_2C_2\) of the boundary curve of Fig. 2 become very short for large \(s\). Let the coordinates of \(C_i\) in Fig. 2 be \(x_0, y_0\). The values of \(q_0, y_0\) are listed in Table I. For \(s \geq 0.7\), \(y_0\) and \(y_0/x_0\) are calculated from the following relation obtained by expansion:

\[
\log_2 q_0 = \log_2 (1/s) + 1 - F(s, q_m)/s
\]

where

\[
F(s, q_m) = \left[ s \log_2 \left( 1 - s q_m + s q_m / s \right) - h(s q_m) + q_m h(s) \right] / q_m
\]

\[
q_m = \log_2 (1/s) + 1 - 2 q_m h(s)/s
\]

\[
y_0 = q_1 \left[ q_1 \log_2 (1/q_1) + (q_1 - q_1 s) \log_2 (1/s) \right].
\]

The table shows that the \(B_1C_1\) and \(B_2C_2\) segments of the boundary diminish very rapidly for increasing \(s\), and \(G_J\) is practically coincident with the capacity region \(G\) for \(s\) larger than 0.7.

We have shown that, for a particular discrete two-user channel, \(G_J\) is a good approximation to \(G\) for strong interference and is practically coincident with \(G\) for very strong interference.

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**The Feedback Capacity of Degraded Broadcast Channels**

ABBAS EL GAMAL

Abstract—The fact that the capacity region of the discrete memoryless physically degraded broadcast channel is not increased by feedback is established.

I. INTRODUCTION

The capacity region of the discrete memoryless degraded broadcast channel (Cover [1]) was established in [2]-[4] and [6]. Bergmans [2] exhibited an achievable rate region. A converse for the binary symmetric broadcast channel was established by Wyner and Ziv [3]; Gallager [4] then proved a converse for the general discrete memoryless degraded broadcast channel. An alternative proof of the converse was given by Ahlswede [6]. Using methods similar to those in [4], it will be shown that the capacity region is unchanged by feedback when the degradation is physical.

II. PRELIMINARIES AND DEFINITIONS

The model under investigation is shown in Fig. 1. There are two sources, the first producing an integer \(W_1 \in \mathbb{W}_1 = \{1, \ldots, M_1\}\), and the second an integer \(W_2 \in \mathbb{W}_2 = \{1, \ldots, M_2\}\). At the \(n\)th transmission the encoder maps the pair \(W_1, W_2\) and the past outputs \(Z_{1,1}, \ldots, Z_{n-1}\) into \(X_n\). Thus \(X_n = f_n(W_1, W_2, Z_{1,1}, \ldots, Z_{n-1}, Y_{1,1}, \ldots, Y_{n-1})\) into \(X_n\). Thus

\[
x_n = f_n(W_1, W_2, Z_{1,1}, \ldots, Z_{n-1}, Y_{1,1}, \ldots, Y_{n-1}), \quad n = 1, 2, \ldots, N.
\]

(1)

The channel consists of three finite alphabets \(x \in \mathbb{X} = \{1, \ldots, J\}, y \in \mathbb{Y} = \{1, \ldots, J\}, z \in \mathbb{Z} = \{1, \ldots, K\}\), and two transition matrices \(P_x(y|x)\) and \(P_{(z|x)}\). By the discrete memorylessness of the channel, for any \(N\)

\[
p(y, z|x) = \prod_{i=1}^{N} p_x(y_i|x_i) p_{(z|x)}(z_i|x_i).
\]

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