Type 2 representation and reasoning for CWW

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Abstract

Computing with words (CWW) is enriched by Type 2 fuzziness. Type 2 fuzziness exists and provides a richer knowledge representation and approximate reasoning for computing with words. First, it has been shown that membership functions, whether (1) they are obtained by subjective measurement experiments, such as direct or reverse rating procedures which captures varying degrees of membership and hence varying meanings of words or else (2) they are obtained with the application of modified fuzzy clustering methods, where they all reveal a scatter plot, which captures varying degrees of meaning for words in a fuzzy cluster. Secondly, it has been shown that the combination of linguistic values with linguistic operators, “AND”, “OR”, “IMP”, etc., as opposed to crisp connectives that are known as t-norms and t-conorms and standard negation, lead to the generation of Fuzzy Disjunctive and Conjunctive Canonical Forms, FDCF and FCCF, respectively. In this paper, we first discuss how one captures Type 2 representation. Then we concentrate on Type 2 reasoning that rests on Type 1 representation. Next, we show how one computes Type 2 reasoning starting with Type 2 representation. It is to be forecasted that in the new millennium more and more researchers will attempt to capture Type 2 representation and develop reasoning with Type 2 formulas that reveal the rich information content available in information granules, as well as expose the risk associated with the graded representation of words and computing with words. This will entail more realistic system model developments, which will help explore computing with perceptions, and computing with words by exposing graded flexibility as well as uncertainty embedded in meaning representation. Crown Copyright © 2002 Published by Elsevier Science B.V. All rights reserved.

1. Introduction

In most current investigations, fuzzy set representations and their logical combinations are based on Type 1 schema for both the knowledge representation and approximate reasoning. First, Type 1 representation is a “reductionist” approach for it discards the spread of membership values by averaging or curve fitting techniques and hence, camouflages the “uncertainty” embedded in the spread of membership values. Therefore, Type 1 representation does not provide a good approximation to meaning representation of words and does not allow computing with words a richer platform. Secondly, Type 1 approximate reasoning relies just on the fuzzified version of the “shortest” forms of the classical Boolean Normal Form formulas, with the “assumption” that the linguistic “AND” corresponds to a t-norm and linguistic “OR” corresponds to a t-conorm in a one-to-one mapping. In Type 1 approximate reasoning, “AND”ness is simply mapped to disjunctive normal form (DNF), and “OR”ness is
simply mapped to conjunctive normal form (CNF), etc. This is again a reductionist, as well as, “myopic” approach to approximate reasoning. It is an old habit from two-valued to fuzzy-valued theory. At least two intuitive expectations come to mind in the light of the limitations of Type 1 representation and reasoning outlined above.

First, intuitively, it is natural to expect that a combination of two imprecise Type 1 memberships should produce new membership functions that capture the compounded increase of imprecision, i.e., produce a Type 2 membership function. Thus, one needs to develop formulas that represent Type 2 uncertainty that arises out of the combination of Type 1 memberships with linguistic connectives, “AND”, “OR”, “IMP”, etc.

Second, again intuitively, we would expect that the combination of two Type 2 memberships should produce new membership functions that capture the compounded increase of imprecision realizable with additional granulation of information.

2. Sources of Type 2 fuzziness

In a historical perspective, it is to be noted that, Zadeh [49] outlined the importance of interval-valued (a special case of Type 2) fuzzy sets in decision processes. Furthermore, Zadeh [50] discussed Type 2 fuzzy set representation and its potential in approximate reasoning. A number of researchers began to investigate Type 2 fuzzy sets and their properties after Zadeh’s introduction of this topic. Some of these are as follows: Mizumoto and Tanaka [23] (1981); Niemien [24]; Yager [47]; Hisdal [12]; Wagenknecht and Hartmann [46]; Roy and Biswas [30]; Burillo and Bustince [5,6]; Bustince and Burillo [7]; John [13]; John et al. [14]; Karniak [15]; Karniak and Mendel [16,17]; Liang and Mendel [19,20].

There are two aspects of Type 2 fuzziness in these investigations: First, it is not explicitly stated how the Type 2 fuzzy set memberships are acquired with the exception of (i) John et al. [14], which indicates that Type 2 fuzzy sets are obtained via neuro-fuzzy clustering of radiographic images and (ii) Mendel et al., which indicate that they are computed by mean μ, and σ of the scatter points obtained from questionnaires. Secondly, most of these works discuss axiomatic properties and related approximate reasoning results, and use the myopic formulas of Type-1 reasoning in developing inference schemas, e.g., Mendel et al., i.e., they do not consider FDCF and FCCF formulas in reasoning schemas.

In contrast, in our investigations, we have dealt with interval-valued Type 2 representation and reasoning starting in the early 80s. First, in experimental studies of measurement of membership functions, it was shown by Norwich and Türkşen [25–27] that direct measurement experiments, conducted in order to extract membership functions for “tall” men, “pleasing” houses, reveal a Type 2 membership representation that can be identified with the mean and spread of scatter points, with standard deviations (Figs. 1 and 2). It should be observed that in these graphs, the spread of scatter points come from the same subject. It is natural to conjecture that the spread would be larger if the scatter points come from a number of subjects in a measurement experiment. It should be noted that the same individual when asked in an experiment gives different membership values to the same height of an individual. The replicated experiments are designed so that there is no memory. Type 2 fuzziness is a natural outcome of the measurement theory. Since individuals give different membership values, the spread of membership values reflects on the one hand the uncertainty
of searching the optimal \((m, c)\) pair \((m^*, c^*)\), we can identify a set of \(m\)'s, \(\{m\}\) that minimize a \(c\), say \(c^*\). Thus having identified \(c^*\), we can then choose a lower value, \(m^L = \min_m \{m, c^*\}\) and an upper value \(m^U = \max_m \{m, c^*\}\).

Thus, for different levels of fuzziness, i.e., order of fuzzy overlaps, \(m\)'s, we get a scatter plot that gives us a ground for Type 2 representation. This means that we can acquire Type 2 membership functions in inductive, unsupervised, learning experiments with training data (Fig. 3).

Before we continue, it should be noted that there are essentially two types of Type 2 fuzziness: (i) interval-valued Type 2 and (ii) full Type 2.

(i) Interval-valued Type 2 fuzziness is a special Type 2 fuzziness where the upper and lower bounds of membership are identified and the spread of membership, either fuzzy or probabilistic and where the spread of membership distribution is ignored with the assumption that membership values between upper and lower values are uniformly distributed or scattered with a membership value of “1” on the \(\mu(\mu(-))\) axis as shown in Fig. 4. Thus, the upper and lower bounds of interval-valued Type 2 fuzziness specify the range of uncertainty about the membership values.

(ii) Full Type 2 fuzziness identifies upper and lower membership values as well as the spread of

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**Fig. 2.** Direct rating for “pleasing” house (subject #1).

**Fig. 3.** Direct rating for “pleasing” house (subject #1).

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**Fig. 2.** Direct rating for “pleasing” house (subject #1).

**Fig. 3.** Type 2 fuzzy sets: (a) a set of membership functions of a cluster; (b) interval-valued membership functions of a cluster.
membership values between these bounds either probabilistically or fuzzily. That is there is a probabilistic or possibilistic distribution of membership values that are between upper and lower bound of membership values in the $\mu(\mu(\cdot))$ axis as shown in Fig. 5. Thus, we obtain a graded distribution of uncertainty between the bounds.

3. Interval-valued fuzzy sets

In conjunction with Type 2 fuzziness found in measurement experiments, interval-valued fuzzy sets were discovered in combination of concepts and hence approximate reasoning by Türksen [32–35] where it was shown that disjunctive and conjunctive normal forms (DNF) and (CNF) of Boolean logic are no longer equivalent due to the fact that the law of excluded middle (LEM), and its dual the law of contradiction (LC) do not hold in fuzzy set and logic theory which were relaxed by Zadeh [48] in his seminal paper.

At the same time, it was pointed out that linguistic operators, “AND”, “OR”, etc., are fuzzy operators and they do not directly correspond to one of the t-norms and t-conorms, which are crisp operators. Linguistic operators “AND”, “OR”, etc., being inherently fuzzy, need to be interpreted in a graded manner, whereas t-norms, $\Delta$’s, and t-conorms, $\lor$’s, are crisp operators that combine singleton fuzzy degrees of information granules and give a singleton fuzzy degree, i.e., $\mu_1 \Delta \mu_2 = \mu_3$ or $\mu_1 \lor \mu_2 = \mu_4$.

In fact, Zimmerman and Zysno [53] had shown that human use of “AND”, “OR” do not directly correspond to a t-norm, such as min, or algebraic product and to a t-conorm, such as max or algebraic sum, respectively. They had proposed “Compensatory ‘AND’”, which either linearly or exponentially combines a t-norm and a t-conorm, such as min–max, or algebraic product and sum with a compensation operator, $\gamma$; i.e.,

(i) Convex linear compensation:

$$F_1(\mu_A, \mu_B; \gamma_1) = \gamma_1 (\mu_A \land \mu_B) + (1 - \gamma_1) (\mu_A \lor \mu_B)$$

or

(ii) Exponential compensation:

$$F_2(\mu_A, \mu_B; \gamma_2) = (\mu_A \land \mu_B)^{\gamma_2} (\mu_A \lor \mu_B)^{1-\gamma_2},$$

where $0 \leq \gamma_1, \gamma_2 \leq 1$.

Later, Türksen [37] showed that in fact $\gamma$ operators of Zimmermann and Zysno [53] were compensation weights that combined DNF and CNF values
of “AND”ness, “OR”ness that Türkşen discovered in 1986. That is, the degrees of “AND”ness should be captured, for the case of complex linear compensation as

\[ F_{\text{LAND}}(\mu_A, \mu_B, \gamma_{\text{LAND}}) = \gamma_{\text{LAND}} \mu_{\text{DNF}}(A \text{ AND } B) + (1 - \gamma_{\text{LAND}}) \mu_{\text{CNF}}(A \text{ AND } B), \]

where “LAND” stands for linear ‘AND’ compensation; the degrees of “OR”ness are captured as

\[ F_{\text{LOR}}(\mu_A, \mu_B, \gamma_{\text{LOR}}) = \gamma_{\text{LOR}} \mu_{\text{DNF}}(A \text{ OR } B) + (1 - \gamma_{\text{LOR}}) \mu_{\text{CNF}}(A \text{ OR } B), \]

where “LOR” stands for linear ‘OR’ compensation; and the degrees of “AND/OR”ness are captured as

\[ F_{\text{LAND OR}}(\mu_A, \mu_B, \gamma_{\text{LAND OR}}) = \gamma_{\text{LAND OR}} \mu_{\text{CNF}}(A \text{ AND } B) + (1 - \gamma_{\text{LAND OR}}) \mu_{\text{DNF}}(A \text{ OR } B), \]

where “LAND OR” stands for linear ‘AND’ compensation; the degrees of “OR”ness are captured as

\[ F_{\text{EAND}}(\mu_A, \mu_B, \gamma_{\text{EAND}}) = (\mu_{\text{DNF}}(A \text{ AND } B))^{\gamma_{\text{EAND}}} (\mu_{\text{CNF}}(A \text{ AND } B))^{1 - \gamma_{\text{EAND}}}, \]

where ‘EAND’ stands for exponential ‘AND’ compensation; the degrees of “OR”ness are captured as

\[ F_{\text{EOR}}(\mu_A, \mu_B, \gamma_{\text{EOR}}) = (\mu_{\text{DNF}}(A \text{ OR } B))^{\gamma_{\text{EOR}}} (\mu_{\text{CNF}}(A \text{ OR } B))^{1 - \gamma_{\text{EOR}}}, \]

where ‘EOR’ stands for exponential ‘OR’ compensation; and the degrees of “AND/OR”ness are captured as

\[ F_{\text{EAND OR}}(\mu_A, \mu_B, \gamma_{\text{EAND OR}}) = (\mu_{\text{CNF}}(A \text{ AND } B))^{\gamma_{\text{EAND OR}}} (\mu_{\text{DNF}}(A \text{ OR } B))^{1 - \gamma_{\text{EAND OR}}}, \]

where ‘E(AND/OR)” stands for exponential ‘AND/OR’ compensation.

These investigations were conducted within the spirit of the “reductionist” approach and therefore reduced Type-2 uncertainty generated by DNF and CNF formulae to Type-1 membership functions with the assignment of \( \gamma \) operators. In Fig. 6, “exponential compensatory ‘AND’” is shown for \( \mu_a = 0.8 \), and \( \mu_b = 0.6 \).

But there remained a fundamental question that needed to be answered from a theoretical point of view. It was known that Boolean formulae, DNF \((\cdot) \equiv (\cdot) \text{ CNF } (\cdot)\) for all 16 possible combinations of any two two-valued sets. But, it was shown that Boolean DNF and CNF formulae are no longer equivalent, when used with fuzzy membership values [35].

That is, they give results such that DNF \((\cdot) \neq \text{ CNF } (\cdot)\), where \((\cdot)\) stands for any of the sixteen combinations of any two linguistic values of any two linguistic variables (see Table 1). In fact, it is also shown that DNF \( \subseteq \text{ CNF} \) for the well-known cases of t-norm and t-conorm based De Morgan Triples formed with standard negation such as \((\wedge, \vee, -)\), \((\otimes, \oplus, -)\), \((L_{\otimes}, L_{\oplus}, -)\), \((D_{\otimes}, D_{\oplus}, -)\), where \((\wedge, \vee)\) stand for min–max, \((\otimes, \oplus)\) stand for algebraic product and sum, \((L_{\otimes}, L_{\oplus})\) stand for Lukasiewicz intersection and union, and \((D_{\otimes}, D_{\oplus})\) stand for drastic intersection and union operators [40].

The fundamental question at this juncture was whether or not fuzzy disjunctive and conjunctive canonical forms can be derived from fuzzy truth tables. In 1970, it was argued that fuzzy truth tables could not be formed [21] for it would require a table with infinite number of rows. In a series of papers [38–42], it was progressively shown that fuzzy truth tables can be constructed and fuzzy disjunctive and conjunctive canonical forms (FDCF) and (FCCF) can be derived from the fuzzy truth tables. More recently, Resconi and Türkşen [29] have shown that there are deeper reasons why FDCF and FCCF are realized.

Along the way, other researchers began to investigate such topics “AND/OR” intervals of possible degrees of belief [54], and Fuzzy normal forms [8,9].

4. FDCF and FCCF expressions

The application of the normal form derivation algorithm [see Appendix A] gives us the derivation of FDCF and FCCF expressions for “A AND B”, shown in row 6 of Table 1, from the fuzzy truth table.
Table 5, as

\[
\text{FDCF (} A \text{ AND } B \text{)} = (A \cap B), A \supseteq B \text{ XOR } A \subset B \text{ is } T,
\]

\[
\text{FCCF (} A \text{ AND } B \text{)} = (A \cup B) \cap (B \cup c(A)) \cap (c(B) \cup A),
\]

\[
A \supseteq B \text{ XOR } A \subset B \text{ is } T
\]

for all t-norm, t-conorm and standard complement-based De Morgan Triples that correspond to symbolic set operation “\(\cap\)”, “\(\cup\)”, “\(\complement\)” in the numerical domain, respectively.

As well, with an appropriate modification of the last column of Table 5, we can drive FDCF and FCCF expressions shown in Table 2 for all the remaining 15 meta-linguistic combinations shown in Table 1. For example for “A OR B”, we get

\[
\text{FDCF (} A \text{ OR } B \text{)} = (A \cap B) \cup (A \cap c(B)) \cup (c(A) \cap B),
\]

\[
A \supseteq B, \text{ XOR } A \subset B \text{ is } T,
\]

\[
\text{FCCF (} A \text{ OR } B \text{)} = (A \cup B), A \supseteq B \text{ XOR } A \subset B \text{ is } T.
\]
For another example, for “A IMPLIES B”, \((A \rightarrow B)\), we get

**FDCF** \((A \rightarrow B)\)

\[
= (A \cap B) \cup (c(A) \cap B) \cup (c(A) \cap c(B)),
\]

\(A \subseteq B, \text{ XOR } A \subset B \text{ is } T,\)

**FCCF** \((A \rightarrow B)\)

\[
= (c(A) \cup B), A \supseteq B, \text{ XOR } A \subset B \text{ is } T.
\]

Again these formulae hold, true for all t-norm, t-conorm and standard complementation operations in the numerical domain that correspond to symbolic set operations “\(\cap\)”, “\(\cup\)”, and “\(c\)”, respectively.

It should be noted that for pseudo t-norm, t-conorm and standard complementation, we need to abide by the order of \(A \supseteq B \text{ XOR } A \subset B\) and adjust all the formulae accordingly, since they are not commutative and associative.

### 5. Impact of FDCF and FCCF

The discovery that we ought to use FDCF and FCCF in fuzzy combination of linguistic concepts and hence approximate reasoning, opens up at least three avenues for the formation of the combination of linguistic con-

cepts and hence approximate reasoning as possible approaches to computing with words:

(i) If we start out with Type 1 membership functions in knowledge representation, say two fuzzy sets A and B, then the combination of these two Type 1 fuzzy sets produce Type 2 fuzzy sets with FDCF \(\subseteq\) FCCF. In Fig. 7, an example of linguistic “OR” combination of two Type 1 fuzzy membership values are shown with \(\mu_{\text{FDCF}}(A \text{ OR } B)(\mu_1, \mu_2)\) and \(\mu_{\text{FCCF}}(A \text{ OR } B)(\mu_1, \mu_2)\).
Fig. 7. $\mu(\mu_1, \mu_2)$ for $\langle \oplus, \otimes, - \rangle$ OR combination of two Type 1 memberships say $\mu_1 = \mu_A(x)$, and $\mu_2 = \mu_B(y)$ where $\mu_1, \mu_2 \in [0, 1]$, $x \in X$, $y \in Y$, and $A, B$ two fuzzy sets.

Furthermore, combinations of more than two fuzzy sets generate $2^{N-1}$ membership functions in a binary tree, where $N$ is the member of linguistic terms. For example, for the combination of three fuzzy sets with Type 1 membership functions, we have, $2^{3-1} = 4$ membership functions:

(1) $\text{FDCF}(\text{FDCF}(\mu_1, \mu_2), \mu_3)$,
(2) $\text{FDCF}(\text{FCCF}(\mu_1, \mu_2), \mu_3)$,
(3) $\text{FCCF}(\text{FDCF}(\mu_1, \mu_2), \mu_3)$,
(4) $\text{FCCF}(\text{FCCF}(\mu_1, \mu_2), \mu_3)$, etc.

As it should be appreciated, in this case, when there are more than two fuzzy sets with Type 1 membership functions, the consequent of the combined linguistic terms approximately becomes Full Type 2 fuzzy sets.

(ii) If we start out with interval-valued Type 2 fuzzy sets in knowledge representation, say two fuzzy sets, $A$ and $B$ then the combination of these two interval-valued Type 2 fuzzy sets, an approximate Full Type 2 fuzzy set with eight elements is obtained as follows:

(1) $\text{FDCF}(\mu_{1L}, \mu_{2L})$, (2) $\text{FDCF}(\mu_{1L}, \mu_{2U})$, (3) $\text{FDCF}(\mu_{1U}, \mu_{2L})$, (4) $\text{FDCF}(\mu_{1U}, \mu_{2U})$, (5) $\text{FCCF}(\mu_{1L}, \mu_{2L})$, (6) $\text{FCCF}(\mu_{1L}, \mu_{2U})$, (7) $\text{FCCF}(\mu_{1U}, \mu_{2L})$, (8) $\text{FCCF}(\mu_{1U}, \mu_{2U})$, where the pair $(\mu_{1L}, \mu_{1U})$ defines the interval-valued Type 2 fuzzy set, say, $A$, and $(\mu_{2L}, \mu_{2U})$ defines the interval-valued Type 2 fuzzy set, say, $B$. Naturally, the combination of more than two interval-valued fuzzy sets generate $2^{N-1} \cdot 2^N$ membership values, where $N$ is the number of linguistic terms.

(iii) If we start out with Full Type 2 fuzzy sets in knowledge representation, say for two fuzzy sets, $A$ and $B$, then the combination of these two Full Type 2 fuzzy sets produce Type 3 fuzzy sets with $\text{FDCF} \neq \text{FCCF}$ computed on $\mu(\mu_1)$ and $\mu(\mu_2)$. Con-
sequently, we would obtain \( \mu_{FDCF}(\mu(\mu_1), \mu(\mu_2)) \) and \( \mu_{FCCF}(\mu(\mu_1), \mu(\mu_2)) \) as Type 3 fuzzy sets for all points \( \mu(\mu_1) \in [0, 1] \) and \( \mu(\mu_2) \in [0, 1] \).

It should be noted that Karniak and Mendel [16,17], Luang and Mendel [20] only compute \( \mu_{FDCF}(\mu(\mu_1), \mu(\mu_2)) \) and ignore \( \mu_{FCCF}(\mu(\mu_1), \mu(\mu_2)) \) in computing “AND”ness, etc. On the other hand, they compute \( \mu_{FCCF}(\mu(\mu_1), \mu(\mu_2)) \), but ignore \( \mu_{FDCF}(\mu(\mu_1), \mu(\mu_2)) \) in computing “OR”ness. That is, they use the “reductionist”,(myopic) formulas, i.e., short form of the Boolean formulas for “AND”ness and “OR”ness.

It is clear that Type 1–Type 2 and Type 2–Type 3 membership generation created by FDCF and FCCF formulas create computational complexity. One way to simplify these, after the application of FDCF and FCCF between any two fuzzy sets with either Type 1 or interval-valued Type 2 or Full Type 2 membership functions, we may discard the values generated between upper and lower membership values generated in \( \mu(\mu) \) space for Type 2 generation and \( \mu(\mu(\mu)) \) space for Type 3 generation, and hence reduce the computational complexity to the computation of interval-valued Type 2 and interval-valued Type 3 membership generation, if and when, we cannot afford extended computing time and effort. For some applications, this may be justified. A skeleton of Type 3 membership generation is shown in Fig. 8.
5.1. Type-2 knowledge representation

There are two approaches to obtain Type-2 knowledge representation: (i) Membership measurement experiments conducted with experts and (2) Fuzzy cluster experiments conducted with FCM algorithm.

**Measurement of membership.** Membership measurement experiments conducted with experts, i.e., subjects, expose the need to represent knowledge with Type-2 fuzzy sets as depicted in Figs. 1 and 2. We have presented various aspects of measurement of membership both theoretically and experimentally [1–4,25–28,31–33,36,43].

Briefly, first it is worthwhile to note that measurement theory deals with representation and uniqueness from empirical relational domain to numerical relational domain. A particular interest in this regard is the scale strength of the numerical representation depending on which axioms of measurement theory can be validated for a particular experimental data obtained from experts. In this regard, it is to be realized that various representation theorems associated with various quantitative structures generate algebraic representation from ordinal to absolute scale strengths (Fig. 9).

Secondly, issues of the scale strength aside, analysts generally get a scatter plot of expert responses. In order to capture variability of these responses, and their information granules, one needs to represent membership values in Type-2 schemas (Figs. 1–5).

This becomes particularly important from the perspective of computing with perceptions and words paradigm [51,52]. In this perspective it is essential that the meaning of words, linguistic terms of linguistic variables are represented effectively with membership grades of the membership, i.e., variation of membership values.

Type 2 representation exposes the variation of membership values and causes a more richer and robust meaning representation of words and their computation.

**Fuzzy clustering.** In applications of fuzzy clustering algorithm, FCM, a major concern is the selection

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**Fig. 9.** Various representation theorems for membership functions.

<table>
<thead>
<tr>
<th>Legend</th>
</tr>
</thead>
<tbody>
<tr>
<td>WO: Weak Ordering</td>
</tr>
<tr>
<td>WA: Weak Associativity</td>
</tr>
<tr>
<td>H: Homogeneity</td>
</tr>
<tr>
<td>Id: Identity</td>
</tr>
<tr>
<td>M: Monotonicity</td>
</tr>
<tr>
<td>Ip: Idempotence</td>
</tr>
<tr>
<td>B: Boundary</td>
</tr>
<tr>
<td>C: Continuity</td>
</tr>
<tr>
<td>Ar: Archimedean</td>
</tr>
<tr>
<td>Sv: Solvability</td>
</tr>
</tbody>
</table>

Algebraic Representation

- Ordinal $< [0,1], \geq S >$
- Ordinal $< [0,1], \geq \text{max} >$
- Ratio $< [0,1], \geq S >$
- Absolute $< [0,1], \geq S_w >$
The determination of Type-2 fuzziness with FCM algorithm also settles the scale strength issue raised in measurement theory. That is, if input data that is used in FCM algorithm is on the absolute scale, then the resulting membership values in Type-2 representation turn out to be on the absolute scale due to the computations required in the FCM algorithm.

5.2. Type 2-approximate reasoning

Type 2 approximate reasoning can be computed in two ways as follows: (1) Interval-valued Type 2 reasoning where we reduce the Type 2 uncertainty by computing FDCF and FCCF formulas only for the determination of upper and lower values of Type 2 memberships; and (2) Full Type 2 reasoning where all available grades between upper and lower membership values are combined in a pairwise combination.

The first approach is partially reductionist, but computationally efficient in comparison to “Full” Type 2 computations. Whereas, the second approach leads to computational complexity of a higher order as discussed previously. In Table 2, the FDCF and FCCF formulas are shown. A detailed discussion and derivation of FDCF and FCCF may be found in Türeşen [40]. Here, we briefly review the derivation of FDCF and FCCF formulae.

In Table 1, meta-linguistic expressions of combined concepts are shown for any two linguistic concepts $A$ and $B$ whether they are represented by two or infinite (fuzzy) valued sets. In Table 2, classical and fuzzy canonical forms are shown under the headings of fuzzy disjunctive canonical forms, FDCF/disjunctive normal forms, DNF, and fuzzy conjunctive canonical forms, FCCF/conjunctive normal forms, CNF, for fuzzy/classical logic, respectively. The derivation of classical normal forms are based on Classical Truth Table, Table 3. The equivalence of classical formula, i.e., $\text{DNF}(\cdot) \equiv \text{CNF}(\cdot)$ for all the 16 possible combinations can be shown to hold with the axioms of classical set and logic theory operations. It should be noted that the set operations, $\cap$, $\cup$, are implicitly assumed to be two-valued, i.e., in $\{0, 1\}$, whereas in Table 3, their truthfullness are explicitly shown to be two-valued, i.e., $\{T, F\}$. Also, in Table 4, “Axioms of classical set and logic operations”, the set operations, $\cap$, $\cup$, are again implicitly assumed to be two-valued. It should be pointed out that in Tables 1–4 there is no explicit
Table 3
Classical truth table interpretations of “A AND B”

<table>
<thead>
<tr>
<th>Truth assignments to classical meta-linguistic variables</th>
<th>Truth assignments to the meta-linguistic expression</th>
<th>Primary conjunctions</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>B</td>
<td>“A AND B”</td>
</tr>
<tr>
<td>T(A)</td>
<td>T(B)</td>
<td>T(A AND B)</td>
</tr>
<tr>
<td>F(A)</td>
<td>T(B)</td>
<td>F(A AND B)</td>
</tr>
<tr>
<td>F(A)</td>
<td>F(B)</td>
<td>F(A AND B)</td>
</tr>
</tbody>
</table>

Table 4
Axioms of classical and set and logic operations

<table>
<thead>
<tr>
<th>Involution</th>
<th>commutativity</th>
<th>A ∪ A = A</th>
</tr>
</thead>
<tbody>
<tr>
<td>Associativity</td>
<td>A ∩ B = B ∩ A</td>
<td></td>
</tr>
<tr>
<td>Distributivity</td>
<td>(A ∪ B) ∩ C = A ∪ (B ∩ C)</td>
<td></td>
</tr>
<tr>
<td>Idempotence</td>
<td>A ∩ A = A</td>
<td></td>
</tr>
<tr>
<td>Absorption</td>
<td>A ∪ (A ∩ B) = A</td>
<td></td>
</tr>
<tr>
<td>Absorption by X and i</td>
<td>A ∪ X = X</td>
<td></td>
</tr>
<tr>
<td>Identity</td>
<td>A ∩ φ = φ</td>
<td></td>
</tr>
<tr>
<td>Law of contradiction</td>
<td>A ∩ c(A) = φ</td>
<td></td>
</tr>
<tr>
<td>Law of excluded middle</td>
<td>A ∩ c(A) = X</td>
<td></td>
</tr>
<tr>
<td>De Morgan’s laws</td>
<td>c(A ∩ B) = c(A) ∪ c(B)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>c(A ∪ B) = c(A) ∩ c(B)</td>
</tr>
</tbody>
</table>

6. Containment vs. equivalence

In classical theory, we have DNF(·) ≡ CNF(·) for all 16 cases shown in Tables 1 and 2. It is shown that FDNF(·) ⊆ CNF(·) for all 16 cases in fuzzy theory for the four well-known de Morgan Triples [35,40]. This is now demonstrated very briefly. For this purpose, we will first show the equivalence of DNF(·) ≡ CNF(·), and then FDNF(·) ⊆ CNF(·).

Table 5
The “fuzzy truth table” of “A AND B” where a, b ∈ [0, 1]

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>A AND B</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1.1)</td>
<td>a ≥ b</td>
<td>T</td>
</tr>
<tr>
<td>(2.1.1)</td>
<td>a ≥ b</td>
<td>T</td>
</tr>
<tr>
<td>(2.1.1)</td>
<td>a ≥ b</td>
<td>T</td>
</tr>
<tr>
<td>(1.1.1)</td>
<td>a ≥ b</td>
<td>T</td>
</tr>
<tr>
<td>(2.1.1)</td>
<td>a ≥ b</td>
<td>T</td>
</tr>
<tr>
<td>(2.1.1)</td>
<td>a &lt; b</td>
<td>F</td>
</tr>
<tr>
<td>(2.2)</td>
<td>a &lt; b</td>
<td>F</td>
</tr>
<tr>
<td>(2.2)</td>
<td>a &lt; b</td>
<td>F</td>
</tr>
<tr>
<td>(2.2)</td>
<td>a &lt; b</td>
<td>F</td>
</tr>
<tr>
<td>(2.2)</td>
<td>a &lt; b</td>
<td>F</td>
</tr>
<tr>
<td>(2.2)</td>
<td>a &lt; b</td>
<td>F</td>
</tr>
<tr>
<td>(2.2)</td>
<td>a &lt; b</td>
<td>F</td>
</tr>
<tr>
<td>(2.2)</td>
<td>a &lt; b</td>
<td>F</td>
</tr>
</tbody>
</table>

Equivalence of DNF and CNF

In classical theory, we have for example:

CNF(A AND B) ≡ DNF(A AND B)
To show this, we start out with the expressions:

\[
\text{DNF}(A \text{ AND } B) = A \cap B \\
\text{CNF}(A \text{ AND } B) = (A \cup B) \cap (A \cup c(B)) \cap (c(A) \cup B)
\]

Next, we apply certain axioms to CNF (\(\cdot\)), as follows:

(idempotency and distributivity) 
\[= [A \cup (B \cap c(B))] \cap [B \cup (A \cap c(A))]\]

(law of contradiction) \[= [A \cup \emptyset] \cap [B \cup \emptyset]\]

(identity, boundary) \[= A \cap B\]

In a similar manner for the other 15 possible combinations of concepts shown in Table 1, it can be shown that DNF(\(\cdot\)) \(\equiv\) CNF(\(\cdot\)) in two-valued set and logic.

**Containment of FCNF and FCCF**

In fuzzy theory, we have

\[\text{FDCF}(A \text{ AND } B) \subseteq \text{FCCF}(A \text{ AND } B)\]

To show this, we start out again with the expressions:

\[\text{FDCF}(A \text{ AND } B) = A \cap B\]

\[\text{FCCF}(A \text{ AND } B) = (A \cup B) \cap (A \cup c(B)) \cap (c(A) \cup B)\]

As indicated, the expressions are the same but in form only. In general, for the class of t-norms and t-conorms and standard complement De Morgan Triples that correspond to the prepositional set operators, “\(\cap\)”, “\(\cup\)”, “\(\cdot\)”, respectively. The axioms of idempotency, distributivity and the law of contradiction, shown in Table 4, are not applicable in fuzzy theory. It is because these axioms are relaxed in general in fuzzy theory that we obtain the containment of FDCF in FCCF.

Next, let us demonstrate this containment relationship as an example for the case of “\(A \text{ OR } B\)” combination and this time in the numerical domain where t-norm, \(A\), and t-conorm, \(\nabla\) operators are applicable.

That is, we need to show that

\[\text{FDCF}(A \text{ OR } B) \subseteq \text{FCCF}(A \text{ OR } B), \text{ i.e.,}\]

\[(A \cap B) \cup (c(A) \cap B) \cup (A \cap c(B)) \subseteq A \cup B\]

In the numerical membership domain, we need to show that

\[\nabla[A(a,b), A(\bar{a},b)] \leq b \quad \text{and} \quad \Lambda(a,b) \leq a\]  

(2)

hold in general where \(\bar{a} = 1 - a\), \(\bar{b} = 1 - b\), and \(a, b \in [0, 1]\). While (2) is true in general for all t-norms, (1) requires a bit of work, but can be shown to hold for the four well-known De Morgan Triples as will be done shortly below. If (1) and (2) hold, it is true that (3) holds.

\[\nabla[\nabla[A(a,b), \Lambda(\bar{a},b)], \Lambda(a,b)] \leq \nabla(a,b).\]  

(3)

Next, we show that (1) holds for well-known cases of t-norms and t-conorms. The following statements are true in general for the class of t-norms and t-conorms. First, it is well known that (4) and (5) are true:

\[aD_L b \leq aL_L b \leq a \otimes b \leq a \land b.\]  

(4)

Next, it is also well known that

\[a \lor b \leq a \oplus b \leq aL_D b \leq aD_D b.\]  

(5)

Consider next the following inequalities, which are all true:

\[\nabla[A(a,b), A(\bar{a},b)] \leq b\]  

(for \(\nabla = \lor\), \(A = \land\), and \(-\));  

(6)

\[\nabla[A(a,b), A(\bar{a},b)] \leq b\]  

(for \(\nabla = \oplus\), \(A = \otimes\), and \(-\));  

(7)

\[\nabla[A(a,b), A(\bar{a},b)] \leq b\]  

(for \(\nabla = \land\), \(A = L_L\), and \(-\));  

(8)

\[\nabla[A(a,b), A(\bar{a},b)] \leq b\]  

(for \(\nabla = \lor\), \(A = D_D\), and \(-\)).  

(9)

In expressions (3), (4), (6), (7), (8), (9), \(D_D, L_L\), \(\oplus\), and \(\land\) are “drastic”, Lukasiewicz, “algebraic” and “minimum” intersection operators, respectively. Also, \(D_D, L_D\), and \(\lor\) are “drastic”, Lukasiewicz, “algebraic”, and “maximum” union operators, respectively.

Since (2) holds in general for all t-norms and conorms then it is clear that for the well-known cases discussed above, we have FDCF (\(A \text{ AND } B\)) \(\subseteq\) FCCF (\(A \text{ AND } B\))!
Also, it can be shown that this containment holds true for all 16 combinations of concepts shown in Table 1 for the well known-cases of t-norms and t-conorms discussed above. It is also clear that special classes of t-norm and t-conorms that are transformed to one of these well-known t-norms and t-conorms also have the containment relationship. However, a general proof for all t-norms and conorms have not been shown as yet. This is left as an open question for researchers!

7. Comprimized reasoning

Yager and Filev (1994) proposed “Comprimized Reasoning” as the convex linear combination of the two extreme reasoning methods that are used in the current literature as \( F(y) = (1 - \beta)F_M(y) + \beta F_G(y) \) where \( F_M(y) \) is the consequence of Mamdani-type reasoning and \( F_G(y) \) is the consequence of GMP-type reasoning. It should be recalled that in Mamdani-type product reasoning, the implication \( A \rightarrow B = A \times B \) and in GMP-reasoning, the implication \( A \rightarrow B = c(A) \cup B \) are used.

Naturally, for a singleton observation \( A'(x) = 1 \) for \( x = x^0 \), we get

\[
A'(x^0) \circ (A(x) \rightarrow B(y)) = A(x^0) \Delta B(y)
\]

in Mamdani-type reasoning, and

\[
A'(x^0) \circ (A(x) \rightarrow B(y)) = c(A(x^0)) \nabla B(y)
\]

in GMP-type reasoning.

We observe that

\[
\text{FDCF} (A \land B) = A \cap B,
\]

\[
\text{FCCF} (A \rightarrow B) = c(A) \cup B.
\]

Therefore, we can write

\[
A(x^0) \Delta B(y) \leq (1 - \beta)[A(x^0) \Delta B(y)] + \beta[c(A(x^0)) \nabla B(y)]
\]

\[
\leq c(A(x^0)) \nabla B(y) \quad \text{for} \quad \beta \in [0, 1].
\]

But, we need to show that \( A \cap B \subseteq c(A) \cup B \) in general in order to prove that \( \beta \) values can be found either experimentally or computationally via supervised learning.

8. FDCF(AND) \( \subseteq \) FCCF(\( \rightarrow \))

Let us show that FDCF (\( A \land B \)) is included in FCCF (\( A \rightarrow B \)) in a similar manner such that FDCF (\( A \land B \)) \( \subseteq \) FCCF (\( A \lor B \)) which is shown in Türksen [37]. However, we will show the alternate proof here.

Recall that

\[
\text{FDCF} (A \land B) = A \cap B,
\]

\[
\text{FCCF} (A \rightarrow B) = (A \cup B) \cap (c(A) \cup B) \cap (A \cup c(B)),
\]

\[
\text{FDCF} (A \rightarrow B) = (A \cap B) \cup (c(A) \cap B) \cup (c(A) \cap c(B)),
\]

FCCF (\( A \rightarrow B \)) = \( c(A) \cup B \).

With the general result of subsection previously discussed, it can be shown that

\[
\text{FDCF} (A \land B) \subseteq \text{FCCF} (A \land B)
\]

and

\[
\text{FDCF} (A \rightarrow B) \subseteq \text{FCCF} (A \rightarrow B)
\]

for the four well-known t-norms and t-conorms.

Also, it is clear that

\[
\text{FDCF} (A \land B) \subseteq \text{FCCF} (A \rightarrow B)
\]

for all t-norms and t-conorms.

Next, it is clear that, for some but not all t-norms and t-conorms

\[
\text{FCCF} (A \land B) \subseteq \text{FDCF} (A \rightarrow B)
\]

However, it can be shown that

\[
\text{FDCF} (A \land B) \subseteq \text{FDCF} (A \rightarrow B).
\]

Since,

\[
A \cap B \subseteq (A \cap B) \cup (c(A) \cap B) \cup (c(A) \cap c(B)).
\]

These arguments are graphically shown in Fig. 10. Therefore, the “convex-linear-comprimized-reasoning”

\[
A(x^0) \Delta B(y) \leq (1 - \beta)[A(x^0) \Delta B(y)] + \beta[c(A(x^0)) \nabla B(y)]
\]

\[
\leq c(A(x^0)) \nabla B(y),
\]

(10)
Fig. 10. The lattice diagram showing the containment relationship between FDCF(AND), FCCF(AND) and FDCF(→) and FCCF(→).

\[ \text{FDCF(→)} = (A \cap B) \cup (c(A) \cap B) \cup (c(A) \cap c(B)) \]

\[ \text{FDCF(AND)} = (A \cup B) \cap (c(A) \cup B) \cap (A \cup c(B)) \]

\[ \text{FCCF(AND)} = c(A) \cup B \]

where “Δ” represents a t-norm and “∇” represents a t-conorm.

Therefore, inequality (10) represents all gradations between the two boundaries of Mamdani-type reasoning identified by FDCF (A AND B) and FCCF (A AND B), and the two boundaries of GMP-type reasoning identified by FDCF (A → B) and FCCF (A → B).

This “convex-linear-comprimize-reasoning” is demonstrated graphically in Figs. 11 and 12 for a
particular $A(x)$ and $B(y)$ and for algebraic and bold operators. It is to be observed that the upper and lower bounds of material and product implications are equal to each other for bold operators (Fig. 12).

9. Exponential-comprimized-reasoning

In analogy to the “linear-comprimized”, we next proposed the “exponential-comprimized-reasoning”.
That is,

$$F(y) = [F_M(y)]^{1-\gamma} [F_G(y)]^\gamma.$$  

It can be shown that

$$A(x^0) B(y) \leq [A(x^0) B(y)]^{1-\gamma} [c(A(x^0)) \nabla B(y)]^\gamma \leq c(A(x^0)) \nabla B(y).$$

Since $A \cap B \subseteq c(A) \cup B$ due to the properties of t-norms and co-norms.

The “exponential-comprimized-reasoning” is demonstrated in Figs. 13 and 14 graphically for particular values of $A(x)$ and $B(y)$ and for algebraic and bold operators.

Here, again it should be observed that the upper and lower bounds of material and product implications are equal to each other for bold operators (Fig. 14).

10. Conclusion

It is argued that Type 2 membership functions provide a better and richer grounding for the meaning representation of words and reasoning with robust computations in computing meaning of words in CWW. The initial step in developing “Full” Type 2 system modeling is “Interval-Valued” Type 2 schema.

Also, it is suggested that Type-2 knowledge representation and reasoning add further refinements to fuzzy system modeling. In Type 2 representation, second-order gradation for the meaning of words, linguistic terms of linguistic variables are induced
to add further refinements to meaning representation of words in computing. There are advantages to Type 2 knowledge representation and reasoning because it does not reduce information granulation due to reductionist curve fitting techniques in knowledge representation and it does not reduce the imprecision expansion due to disjunction and conjunctive canonical forms. The exposition of information content embedded in information granules allows decision makers to be more aware of risks associated with imprecise information availability and its analysis.

It was also shown that for lower and upper bounds determined with bold operators, the material implication and the product implication types of reasoning are equivalent to each other. Such results suggest that there is an extra information content to be discovered with Type 2 knowledge representation and computation which is executed with Type 2 reasoning formulas.

This further suggests that if we capture knowledge representation with Type 2 schema and compute with Type 2 reasoning formulas, we can discover additional information content. For example, Full Type 2 reasoning produces Type 3 fuzziness which may be useful in some cases. It should be recalled that Type 1, Type 2 and Type 3 information are analogous to first, second and third moments in statistical calculations but not the same. We expect to report on some of these insights in future investigations.

Appendix A Canonical form derivation algorithm

(a) First, assign truth values \( T, F \) to the meta-linguistic values (labels, variables) \( A \) and \( B \) and then assign truth values \( T, F \) to the meta-linguistic expression of concern, say “\( A \) AND \( B \)” in order to define its meaning.

(b) Next, construct primary conjunctions of the set symbols \( A, B \), corresponding to linguistic values such that in a given row,
Fig. 14. Graph of exponential compromise reasoning $\mu = [A(x^\ast) \otimes B(y)]^{1-\beta} + [c(A(x^\ast)) \oplus B(y)]^\beta$ for $A(x^\ast) = (0.8)$ and $B(y) = (0.6)$ for bold operators.

(i) if a $T$ appears, then take the set affirmation symbol of that meta-linguistic variable; otherwise

(ii) if an $F$ appears, then take the set of complementation symbol of that meta-linguistic variable;

(iii) next, conjunct the two symbols.

(c) Then, construct the disjunctive normal form of the meta-linguistic expression of concern:

(i) first, take the conjunctions corresponding to the $T$’s of the truth assignment made under the column of the meta-linguistic expression, such as “$A$ AND $B$”, and

(ii) next, combine these conjunctions with disjunctions.

(d) Next, construct the conjunctive normal form of the meta-linguistic expression of concern:

(i) first, take the conjunctions corresponding to $F$’s of the truth assignment made under the column of the meta-linguistic expression, such as “$A$ AND $B$” and

(ii) then, combine these conjunctions with disjunctions, and

(iii) next, take the complement of these disjuncted conjunctions.

References


